GENERATION OF RELATIVE COMMUTATOR SUBGROUPS IN CHEVALLEY GROUPS

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Abstract. Let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring and let I, J be two ideals of R. In the present paper we describe generators of the commutator groups of relative elementary subgroups $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ both as normal subgroups of the elementary Chevalley group $E(\Phi,R)$, and as groups. Namely, let $x_\alpha(\xi)$, $\alpha\in\Phi$, $\xi\in R$, be an elementary generator of $E(\Phi,R)$. As a normal subgroup of the absolute elementary group $E(\Phi,R)$, the relative elementary subgroup is generated by $x_\alpha(\xi)$, $\alpha\in\Phi$, $\xi\in I$. Classical results due to Michael Stein, Jacques Tits and Leonid Vaserstein assert that *as a group* $E(\Phi,R,I)$ is generated by $z_\alpha(\xi,\eta)$, where $\alpha\in\Phi$, $\xi\in I$, $\eta\in R$. In the present paper, we prove the following *birelative* analogues of these results. As a normal subgroup of $E(\Phi,R)$ the relative commutator subgroup $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ is generated by the following three types of generators: i) $\big[x_\alpha(\xi),z_\alpha(\zeta,\eta)\big]$, ii) $\big[x_\alpha(\xi),x_{-\alpha}(\zeta)\big]$, and iii) $x_\alpha(\xi\zeta)$, where $\alpha\in\Phi$, $\xi\in I$, $\zeta\in J$, $\eta\in R$. As a group, the generators are essentially the same, only that type iii) should be enlarged to iv) $z_\alpha(\xi\zeta,\eta)$. For classical groups, these results, with much more computational proofs, were established in previous papers by the authors. There is already an amazing application of these results, namely in the recent work of Alexei Stepanov [27] on relative commutator width.

Introduction

Let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring with 1, and let $G(\Phi,R)$ be a Chevalley group of type Φ over R. For the background on Chevalley groups over rings see [32] or [33], where one can find many further references. We fix a split maximal torus $T(\Phi,R)$ in $G(\Phi,R)$ and consider root unipotents $x_\alpha(\xi)$ elementary with respect to $T(\Phi,R)$. The subgroup $E(\Phi,R)$ generated by all $x_\alpha(\xi)$, where $\alpha\in\Phi$, $\xi\in R$, is called the [absolute] elementary subgroup of $G(\Phi,R)$.

Now, let $I\trianglelefteq R$ be an ideal of R. Then the relative elementary subgroup $E(\Phi,R,I)$ is defined as the *normal* subgroup of $E(\Phi,R)$, generated by all elementary root unipotents $x_\alpha(\xi)$ of level I,

$$E(\Phi,R,I)=\big\langle x_\alpha(\xi)\mid \alpha\in\Phi,\ \xi\in R\big\rangle^{E(\Phi,R)}.$$

In other words, as a normal subgroup of $E(\Phi,R)$, the relative elementary subgroup $E(\Phi,R,I)$ is generated by $x_\alpha(\xi)$, where $\alpha\in\Phi$, $\xi\in I$.

A starting point of the present work is the following classical result. Morally, it goes back to the Thesis of Michael Stein [25], which contains all calculations necessary for its proof. In a slightly weaker form it was first stated by Jacques Tits [30]. Namely, Proposition 1 there asserts that $E(\Phi, R, I)$ is generated by its intersections with the fundamental SL_2's. However, the earliest reference for the precise form below, we could trace, is the work by Leonid Vaserstein [31], Theorem 2.

Theorem 1 (Stein—Tits—Vaserstein)**.** *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I be an ideal of a commutative ring R. Then as a group $E(\Phi, R, I)$ is generated by the elements of the form*

$$z_\alpha(\xi, \eta) = x_{-\alpha}(\eta) x_\alpha(\xi) x_{-\alpha}(-\eta),$$

where $\xi \in I$ for $\alpha \in \Phi$, while $\eta \in R$.

In the present paper, we delve a bit deeper into the structure of relative K_1's, proving a *birelative* version of the above result. Namely, we determine elementary generators of the mixed commutator subgroups $[E(\Phi, R, I), E(\Phi, R, J)]$ of two relative elementary subgroups $E(\Phi, R, I)$ and $E(\Phi, R, J)$, corresponding to ideals $I, J \trianglelefteq R$.

It is easy to prove that, apart from the known small exceptions, one has

$$E(\Phi, R, IJ) \leq \big[E(\Phi, R, I), E(\Phi, R, J)\big] \leq G(\Phi, R, IJ),$$

where $G(\Phi, R, I)$ denotes the principal congruence subgroup of $G(\Phi, R)$, of level I, see Theorem 4 below for a precise statement.

Thus, in particular, $[E(\Phi, R, I), E(\Phi, R, J)]$ contains all elements $z_\alpha(\xi\zeta, \eta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$. In some cases, for instance when I and J are comaximal, $I + J = R$, one can prove that

$$[E(\Phi, R, I), E(\Phi, R, J)] = E(\Phi, R, IJ).$$

However, easy examples show that in general the mixed commutator subgroup can be strictly larger than $E(\Phi, R, IJ)$ even for very nice 1-dimensional rings, see [24, 21, 22]. Since $E(\Phi, R, IJ)$ is already normal in $E(\Phi, R)$, one needs *additional* generators to span $[E(\Phi, R, I), E(\Phi, R, J)]$ as a subgroup, and even as a normal subgroup of $E(\Phi, R)$.

In the present paper we display the missing generators. As in the case of the relative elementary subgroups themselves, these generators sit in the fundamental SL_2's and are in fact commutators of *some* elementary generators of $E(\Phi, R, I)$ and $E(\Phi, R, J)$.

Now we are in a position to state the main results of the present paper, which provide such sets of generators of $[E(\Phi, R, I), E(\Phi, R, J)]$ as a normal subgroup of $E(\Phi, R)$, and as a group, respectively.

Theorem 2. *Let Φ be a reduced irreducible root system of rank ≥ 2 and I, J be two ideals of a commutative ring R. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $\theta \in R$ is contained in the ideal $\theta^2 R + 2\theta R$.*

Then as a normal subgroup of the elementary Chevalley group $E(\Phi, R)$, the mixed commutator subgroup $[E(\Phi, R, I), E(\Phi, R, J)]$ is generated by the the elements of the form

- $\big[x_\alpha(\xi), z_\alpha(\zeta, \eta)\big]$,
- $\big[x_\alpha(\xi), x_{-\alpha}(\zeta)\big]$,
- $x_\alpha(\xi\zeta)$,

where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

Modulo level calculations, and the relative standard commutator formula [36, 16] this theorem easily implies the following result.

Theorem 3. *Let Φ be a reduced irreducible root system of rank ≥ 2. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $\theta \in R$ is contained in the ideal $\theta^2 R + 2\theta R$.*

Further, let I and J be two ideals of a commutative ring R. Then the mixed commutator subgroup $\big[E(\Phi, R, I), E(\Phi, R, J)\big]$ is generated as a group by the elements of the form

- $\big[x_\alpha(\xi), z_\alpha(\zeta, \eta)\big]$,
- $\big[x_\alpha(\xi), x_{-\alpha}(\zeta)\big]$,
- $z_\alpha(\xi\zeta, \eta)$,

where in all cases $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

In the proofs, the generating sets, described in these theorems, will be denoted by X and Y, respectively. Then the theorems assert that

$$\big[E(\Phi, R, I), E(\Phi, R, J)\big] = \langle X \rangle^{E(\Phi,R)} = \langle Y \rangle.$$

Clearly, $X \subseteq Y$.

For the case of the general linear group $\mathrm{GL}(n, A)$ over a [not necessarily commutative] associative ring A, a similar set of generators of mixed commutator subgroups as normal subgroups of the absolute elementary group was constructed by the first and the third author in [20]. Of course, in the non-commutative case $[E(n, A, I), E(n, A, J)] \geq E(n, A, IJ + JI)$, so that one has to list both $t_{ij}(\xi\zeta)$ and $t_{ij}(\zeta\xi)$, where $1 \leq i \neq j \leq n$, $\xi \in I$, $\zeta \in J$, as generators. Later, this result was generalised to unitary groups in our paper [17], see also [18] for background, detailed overview, and somewhat finer proofs.

A slightly more delicate construction of generators of the mixed commutator subgroups as subgroups was first carried through in our joint paper with Alexei Stepanov [12], again in the simplest case of $\mathrm{GL}(n, R)$. As before in the non-commutative case one has to engage both $z_{ij}(\xi\zeta, \eta)$ and $z_{ij}(\zeta\xi, \eta)$ as generators.

In the context of Chevalley groups, these results were announced in our joint papers with Alexei Stepanov [10] and [11]. They are instrumental in Stepanov's proof of the bounded width of relative commutators in terms of elementary generators [27].

The paper is organised as follows. In § 1 we recall background facts that will be used in our proofs. In § 2 we implement a slightly stronger version of level calculations, Theorem 4, further elaborating [16], Lemma 17. After that, in § 3 we are in a position to prove Theorem 2, while in § 4 we prove Theorem 3 and derive some of its corollaries. Finally, in § 5 we discuss some further aspects of relative commutator subgroups $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$, describe several cognate results and applications, and formulate some unsolved problems.

1. Some preliminary facts

Here we collect some obvious or well known classical facts, which will be used in our proofs. For background information on Chevalley groups over rings, see [32, 33], where one can find many further references.

1.1. Commutator identities. Let G be a group. For any $x,y\in G$, ${}^xy=xyx^{-1}$ denotes the left x-conjugate of y. As usual, $[x,y]=xyx^{-1}y^{-1}$ denotes the [left normed] commutator of x and y. We will make constant use of the following obvious commutator identities, usually, without any specific reference

(C1) $[x,yz]=[x,y]\cdot{}^y[x,z]$,

(C2) $[xy,z]={}^x[y,z]\cdot[x,z]$,

(C3) $[x,{}^yz]={}^y[{}^{y^{-1}}x,z]$,

(C4) $[{}^yx,z]={}^y[x,{}^{y^{-1}}z]$,

(C5) $[y,x]=[x,y]^{-1}$.

Let $F,H\le G$ be two subgroups of G. By definition, the mixed commutator subgroup $[F,H]$ is the subgroup generated by all commutators $[x,y]$, where $x\in F$, $y\in H$. Clearly, $[F,H]=[H,F]$. A subgroup $H\le G$ is normal in G if $[G,H]\le H$. A group G is called perfect if $[G,G]=G$ and a subgroup $H\le G$ is called G-perfect if $[G,H]=H$.

1.2. Steinberg relations. As in the introduction, we denote by $x_\alpha(\xi)$, $\alpha\in\Phi$, $\xi\in R$ the elementary generators of the [absolute] elementary subgroup $E(\Phi,R)$. All results of the present paper are [directly or indirectly] based on the Steinberg relations among the elementary generators, which will be used without any specific reference.

(R1) Additivity of x_α,

$$x_\alpha(\xi+\eta)=x_\alpha(\xi)x_\alpha(\eta).$$

(R2) Chevalley commutator formula

$$[x_\alpha(\xi), x_\beta(\eta)] = \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi^i\eta^j),$$

where $\alpha \neq -\beta$ and $N_{\alpha\beta ij}$ are the structure constants which do not depend on a and b. Notice, though, that for $\Phi = \mathrm{G}_2$ they may depend on the order of the roots in the product on the right hand side.

In particular, relation (R1) implies that $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$ is a subgroup of $E(\Phi, R)$, isomorphic to R^+, called an [elementary] root subgroup.

1.3. **Congruence subgroups.** Let $\rho_I : R \longrightarrow R/I$ be the reduction modulo I. By functoriality, it defines the group homomorphism $\rho_I : G(\Phi, R) \longrightarrow G(\Phi, R/I)$. The kernel of ρ_I is denoted by $G(\Phi, R, I)$ and is called the principal congruence subgroup of $G(\Phi, R)$ of level I. In turn, the full pre-image of the centre of $G(\Phi, R/I)$ with respect to the reduction homomorphism ρ_I is called the full congruence subgroup of level I, and is denoted by $C(\Phi, R, I)$.

A classical theorem by Andrei Suslin, Vyacheslav Kopeiko and Giovanni Taddei, stated in the next subsection, asserts that $E(\Phi, R, I)$ is a normal subgroup of $G(\Phi, R)$. In particular, the quotient

$$\mathrm{K}_1(\Phi, R, I) = G(\Phi, R, I)/E(\Phi, R, I),$$

is a group and not just a pointed set.

Theorem 2 is purely elementary, in the technical sense that all calculations may take place inside $E(\Phi, R)$ and be performed in terms of the Steinberg relations among elementary generators. However, our proof of Theorem 3 is not elementary, in this sense, and depends on the fact that, modulo the relative elementary subgroup $E(\Phi, R, IJ)$, a mixed commutator subgroup $\big[E(\Phi, R, I), E(\Phi, R, J)\big]$ lives inside $\mathrm{K}_1(\Phi, R, IJ)$.

1.4. **Standard commutator formula.** In the present paper, we use the following fundamental facts, known as the [absolute] standard commutator formulas. The first proof of this result by Giovanni Taddei [29] and Leonid Vaserstein [31] was based on localisation. We refer the reader to [13] for a slightly easier proof, based on similar ideas, and to [14] for many further references, pertaining to the classical cases, and other proofs.

Lemma 1. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then for any ideal* I *of the ring* R *one has the following inclusions*

$$\big[E(\Phi, R, I), G(\Phi, R)\big], \big[G(\Phi, R, I), E(\Phi, R)\big] \leq E(\Phi, R, I).$$

It is classically known, that the only cases, where $E(\Phi, R)$ is not perfect — or, for that matter, where $E(\Phi, R, I)$ is not $E(\Phi, R)$-perfect — are groups of types C_2 and G_2 over a ring having residue fields $\mathbb{F}_2$ of two elements. Thus,

Lemma 2. *Let* $\mathrm{rk}(\Phi) \geq 2$*. In the case* $\Phi = \mathrm{C}_2, \mathrm{G}_2$ *assume additionally that* R *does not have residue fields* $\mathbb{F}_2$ *of two elements. Then for any ideal* I *of the ring* R *one has the following inclusions*

$$\big[E(\Phi, R, I), G(\Phi, R)\big] = \big[G(\Phi, R, I), E(\Phi, R)\big] = E(\Phi, R, I).$$

Actually, in the final section we recall a *birelative* standard commutator formula, which is a simultaneous generalisation of both standard formulas above. This formula, and attempts to generalise it, served as the main motivation behind the results of the present paper. However, it is not directly used in their proofs.

1.5. **Elementary subgroups of level** I**.** Let I be an ideal of R. In the sequel we use also the following non relativised version of elementary subgroups of level I:

$$E(\Phi, I) = \big\langle x_\alpha(\xi) \mid \alpha \in \Phi,\ \xi \in R \big\rangle$$

By definition, $E(\Phi, R, I) = E(\Phi, I)^{E(\Phi,R)}$ is the normal closure of $E(\Phi, I)$ in the absolute elementary group $E(\Phi, R)$, and thus, normal in $G(\Phi, R)$.

Lemma 3. *Let* $\mathrm{rk}(\Phi) \geq 2$ *and further let* I *and* J *be two ideals of* R*. Assume that either* $\Phi \neq \mathrm{C}_l$*, or* $2 \in R^*$*. Then one has* $E(\Phi, R, IJ) \leq E(\Phi, I + J)$*.*

To state an analogue of this inclusion in the exceptional case, we should distinguish the ideal I^2, generated by the products ab, where $a, b \in I$, from the ideal $I^{\boxed{2}}$, generated by a^2, where $a \in I$. Clearly, when $2 \in R^*$ these ideals coincide, but this case is trivial anyway. In general one can only guarantee a weaker inclusion $E(\Phi, R, I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}}) \leq E(\Phi, I + J)$. In fact, both the generic and the exceptional cases are special instances of the following more precise statement,

$$E(\Phi, R, IJ, I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}}) \leq E(\Phi, I + J),$$

see [16], Lemma 16. Here, the left hand side is the relative elementary subgroups corresponding to the admissible pair $(IJ, I^{\boxed{2}}J + 2IJ + IJ^{\boxed{2}})$, see [1, 2], or [8] for further references.

1.6. **Unitriangular subgroups.** Further, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots, respectively.

The unipotent radicals of the opposite Borel subgroups, standard with respect to this order, are defined as follows:

$$U = U(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R \big\rangle,$$

$$U^- = U^-(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi^-,\ \xi \in R \big\rangle.$$

By definition, $E(\Phi, R) = \langle U(\Phi, R), U^-(\Phi, R)\rangle$.

The Chevalley commutator formula immediately implies the following result, see for instance, [5] or [26].

Lemma 4. *The group $U(\Phi, R)$ admits factorisation $U(\Phi, R) = \prod X_\alpha$, over all positive roots $\alpha \in \Phi^+$, taken in any prescribed order. For a given order, an expression of an element $u \in U(\Phi, R)$ in the form $u = \prod x_\alpha(u_\alpha)$, where $u_\alpha \in R$, is unique.*

In turn, uniqueness of factors in Lemma 4 immediately implies the following result, which will be used in the proof of Theorem 2.

Lemma 5. *For any ideal I of R one has*

$$G(\Phi, R, I) \cap U(\Phi, R) \leq E(\Phi, I).$$

It is easy to see this also uses representations of $G(\Phi, R)$. Namely, one considers a non-trivial rational representation $\pi : G(\Phi, R) \longrightarrow \mathrm{GL}(n, R)$ such that $T(\Phi, R)$ is represented by diagonal matrices, $U(\Phi, R)$ is represented by upper unitriangular matrices, and $U^-(\Phi, R)$ is represented by lower unitriangular matrices. If π is faithful, one has $G(\Phi, R, I) = \pi^{-1}(\pi(G(\Phi, R)) \cap \mathrm{GL}(n, R, I))$, see, for instance, [8], Lemma 6. In particular, a product $\prod x_\alpha(\xi_\alpha)$, $\alpha \in \Phi^+$, belongs to $G(\Phi, R, I)$ if and only if $\xi_\alpha \in I$ for all $\alpha \in \Phi^+$.

1.7. **Parabolic subgroups.** The main role in our simplified proof of Theorem 2 is played by the Levi decomposition for [elementary] parabolic subgroups. Classically, it asserts that any parabolic subgroup P of $G(\Phi, R)$ can be expressed as the semi-direct product $P = L_P \leftthreetimes U_P$ of its unipotent radical $U \trianglelefteq P$ and a Levi subgroup $L_P \leq P$. However, we do not wish to recall the requisite background, to state the general case. Luckily, this is not at all nesessary.

• Since we calculate inside $E(\phi, R)$, we can limit ourselves to the *elementary* parabolic subgroups, spanned by some root subgroups X_α.

• Since we can choose the order on Φ arbitrarily, we can further limit ourselves to *standard* parabolic subgroups.

• Since the proof of Theorem 2 easily reduces to groups of rank 2, we could only consider rank 1 parabolic subgroups, which *in this case* coincide with maximal parabolic subgroups.

It is notationally easier to describe rank 1 parabolics, which in discrepancy with the usage of the theory of algebraic groups, will be called *minimal* parabolics. Namely, we fix an r, $1 \leq r \leq l$, and consider the subgroup

$$P_r = \langle U, X_{\alpha_r} \rangle \leq E(\Phi, R),$$

which we call the r-th *elementary* minimal standard parabolic subgroup of $E(\Phi, R)$. Further, set

$$U_r = \prod X_\alpha, \qquad \alpha \in \Phi^+, \quad \alpha \neq \alpha_r;$$

this is the unipotent radical of P_r. Finally, let $L_r = \langle X_{\alpha_r}, X_{-\alpha_r} \rangle$ be the [standard] Levi subgroup of P_r.

The following well result is an immediate corollary of the general Levi decomposition. However, this special instance of Levi decomposition can be easily derived from

the Chevalley commutator formula, without any reference to the theory of algebraic groups.

Lemma 6. *For any $1 \leq r \leq \mathrm{rk}(\Phi)$ the group P_r is the semi-direct product $P_r = L_r \leftthreetimes U_r$ of its unipotent radical $U_r \trianglelefteq P_r$ and its Levi subgroup $L_r \leq P_r$.*

The most important part is the [obvious] claim that U_r is normal in P_r.

2. Levels of mixed commutators

The following easy level calculation was performed on several occasions in various contexts, see for instance, Theorem 1 of Hong You [36] for a slightly weaker statement[1]. In this form, it is essentially a combination of [a slightly stronger version of] Lemma 17 and of Lemma 19 in [16]. The left-most inclusion serves to motivate the statements of Theorems 2 and 3, and to verify that the X and Y are actually contained in $\big[E(\Phi, R, I), E(\Phi, R, J)\big]$, whereas the right-most inclusion will be used to derive Theorem 3 from Theorem 2.

Theorem 4. *Let Φ be a reduced irreducible root system of rank ≥ 2. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $\theta \in R$ is contained in the ideal $\theta^2 R + 2\theta R$.*

Then for any two ideals I and J of the ring R one has the following inclusion

$$E(\Phi, R, IJ) \leq \big[E(\Phi, I), E(\Phi, J)\big] \leq \big[E(\Phi, R, I), E(\Phi, R, J)\big] \\ \leq \big[G(\Phi, R, I), G(\Phi, R, J)\big] \leq G(\Phi, R, IJ).$$

In the cases $\Phi = \mathrm{C}_2$ and $\Phi = \mathrm{G}_2$, the left-most inclusion may fail without these additional assumptions. For rings having residue field $\mathbb{F}_2$, this was observed in connection with Lemma 2. Without additional assumption, in the case $\Phi = \mathrm{C}_2$ one can only claim that

$$E(\Phi, R, IJ, I^{\boxed{2}}J + 2IJ + I^{\boxed{2}}2) \leq \big[E(\Phi, R, I), E(\Phi, R, J)\big],$$

In other words, one has to distinguish the levels of short and long roots, and modify the generating sets in Theorems 2 and 3 accordingly. Since the case of $\mathrm{Sp}(4, R)$ requires a separate analysis anyway, in the present paper we do not pursue this any further, not to overcharge the statements and proofs with such technical details, peculiar for this particular case.

Proof. All inclusions, apart from the left-most one, are either obvious, or established in [16], Lemmas 17 and 19. However, in the proof of [16], Lemma 17, we only demonstrated a weaker inclusion

$$E(\Phi, R, IJ) \leq \big[E(\Phi, R, I), E(\Phi, R, J)\big].$$

[1]With an extra-assumption in the case $\Phi = \mathrm{G}_2$, and a missing assumption in the case of $\Phi = \mathrm{C}_l$, $l \geq 3$.

Thus, it remains to verify that $E(\Phi,R,IJ)$ is contained already in $\big[E(\Phi,I),E(\Phi,J)\big]$. This can be done along the same lines, as in [16], but requires some extra care. For the sake of brevity, in the sequel we denote $\big[E(\Phi,I),E(\Phi,J)\big]$ by H.

Namely, it suffices to show that the generators of the group $E(\Phi,R,IJ)$ belong to H. Indeed, let

$$z_\alpha(\xi\zeta,\eta)={}^{x_{-\alpha}(\eta)}x_\alpha(\xi\zeta),\qquad \xi\in I,\quad \zeta\in J,\quad \eta\in R,$$

be one of these generators. However, in [16] the right hand side of the inclusion was already normal in $E(\Phi,R)$, so that we merely had to verify that $x_\alpha(\xi\zeta)\in H$. Here, at this stage we do not yet know that the right hand side is normal — this is exactly what we are attempting to prove! Thus, now at each step we have to ascertain that all occurring extra factors are actually contained in H.

• First, assume that α can be embedded in a root system of type A_2. Then there exist roots $\beta,\gamma\in\Phi$, of the same length as α such that $\alpha=\beta+\gamma$, and $N_{\beta\gamma11}=1$. By the above, we can express $x_\alpha(\xi\zeta)$ as

$$x_\alpha(\xi\zeta)=[x_\beta(\xi),x_\gamma(\zeta)]\in H.$$

Thus,

$$z_\alpha(\xi\zeta,\eta)={}^{x_{-\alpha}(\eta)}[x_\beta(\xi),x_\gamma(\zeta)]=[{}^{x_{-\alpha}(\eta)}x_\beta(\xi),{}^{x_{-\alpha}(\eta)}x_\gamma(\zeta)]=\\ [x_\beta(\xi)x_{-\gamma}(\pm\xi\eta),x_\gamma(\zeta)x_{-\beta}(\pm\zeta)\eta]\in H,$$

as claimed.

This proves the lemma for simply laced Chevalley groups, and for the Chevalley group of type F_4. It also proves necessary inclusions for *short* roots in Chevalley groups of type C_l, $l\geq 3$, for *long* roots in Chevalley groups of type B_l, $l\geq 3$, and for *long* roots in the Chevalley group of type G_2.

For the remaining cases, the idea of the proof is similar, but requires more care, due to the more complicated form of the Chevalley commutator formula.

• Next, assume that α can be embedded in a root system of type C_2 as a *long* root. We wish to prove that $z_\alpha(\xi\zeta,\eta)\in H$. As the first approximation, we prove that

$$z_\alpha(\xi^2\zeta,\eta),z_\alpha(\xi\zeta^2,\eta)\in H.$$

There exist roots $\beta,\gamma\in\Phi$, such that $\alpha=\beta+2\gamma$ and $N_{\beta\gamma11}=1$. Clearly, in this case β is long and γ is short. Take an arbitrary $\theta\in R$. Then

$$[x_\beta(\theta\xi),x_\gamma(\zeta)]=x_{\beta+\gamma}(\theta\xi\zeta)x_\alpha(\pm\theta\xi\zeta^2)\in H,$$

whereas

$$[x_\beta(\xi),x_\gamma(\theta\zeta)]=x_{\beta+\gamma}(\theta\xi\zeta)x_\alpha(\pm\theta^2\xi\zeta^2)\in H.$$

Comparing these two inclusions we can conclude that

$$\begin{aligned}z_\alpha((\theta^2-\theta)\xi\zeta^2,\eta) &= {}^{x_{-\alpha}(\eta)}[x_\beta(\xi),x_\gamma(\theta\zeta)]\cdot{}^{x_{-\alpha}(\eta)}[x_\gamma(\zeta),x_\beta(\theta\xi)] = \\ &[{}^{x_{-\alpha}(\eta)}x_\beta(\xi),{}^{x_{-\alpha}(\eta)}x_\gamma(\theta\zeta)]\cdot[{}^{x_{-\alpha}(\eta)}x_\gamma(\zeta),{}^{x_{-\alpha}(\eta)}x_\beta(\theta\xi)] = \\ &[x_\beta(\xi),x_\gamma(\theta\zeta)x_{-\beta-\gamma}(\pm\theta\zeta\eta)x_{-\beta}(\pm\eta\theta^2\zeta^2)]\cdot \\ &[x_\gamma(\zeta)x_{-\beta-\gamma}(\pm\zeta\eta)x_{-\beta}(\pm\eta\zeta^2),x_\beta(\theta\xi)]\in H.\end{aligned}$$

By assumption R does not have residue field of 2 elements, and thus the ideal generated by $\theta^2-\theta$, where $\theta\in R$, is not contained in any maximal ideal, and thus coincides with R. Now, since $z_\alpha(\lambda+\mu,\eta)=z_\alpha(\lambda,\eta)z_\alpha(\mu,\eta)$, while I and J are ideals of R, it follows that $z_\alpha(\xi\zeta^2,\eta)\in H$. Interchanging ξ and ζ we see that $z_\alpha(\xi^2\zeta,\eta)\in H$.

Now, we can pull the left bootstrap, before returning again to the right one.

• Namely, assume that α can be embedded in a root system of type C_2 as a *short* root. Choose β and γ subject to the same condition, as in the preceding item. In other words, $\alpha=\beta+\gamma$, β is long, γ is short, and $N_{\beta\gamma11}=1$. Then, clearly

$$z_\alpha(\xi\zeta,\eta)={}^{x_{-\alpha}(\eta)}[x_\beta(\xi),x_\gamma(\zeta)]\cdot{}^{x_{-\alpha}(\eta)}x_{\alpha+\gamma}(\mp\xi\zeta^2).$$

Pulling conjugation by $x_{-\alpha}(\eta)$ inside the first commutator, we get

$$[{}^{x_{-\alpha}(\eta)}x_\beta(\xi),{}^{x_{-\alpha}(\eta)}x_\gamma(\zeta)]=[x_\beta(\xi)x_{-\gamma}(\pm\xi\eta)x_{-\alpha-\gamma}(\pm\xi\eta^2),x_\gamma(\zeta)x_{-\beta}(\pm\zeta\eta)]\in H.$$

On the other hand, from the previous item we already know that $x_{\alpha+\gamma}(\mp\xi\zeta^2)$ can be expressed as a product of *some* commutators of the form $[x_\beta(\rho),x_\gamma(\sigma)]$, where $\rho\in I$, $\sigma\in J$, *provided* that R does not have residue field of 2 elements. Exactly the same calculation as in the above display line shows that the conjugates of these commutators by $x_{-\alpha}(\eta)$ remain inside H. Observe, that from the first item, we already know that for $\Phi=\mathrm{B}_l$, $l\geq 3$, even the stronger inclusion $z_{\alpha+\gamma}(\xi\zeta,\eta)\in H$ holds without any such assumption.

Thus, in both cases we can conclude that $z_\alpha(\xi\zeta,\eta)\in H$ for a *short* root α. Again, already from the first item we know that for $\Phi=\mathrm{C}_l$, $l\geq 3$, this inclusion holds without any assumptions on R.

On the other hand, a *long* root α of $\Phi=\mathrm{C}_l$, $l\geq 3$, cannot be embedded in an irreducible rank 2 subsystem other than C_2. This leaves us with the analysis of exactly two rank 2 cases: $\Phi=\mathrm{C}_2$ and α is long and $\Phi=\mathrm{G}_2$ and α is short. This is where one needs the additional assumptions on R. We pull the right bootstrap once more, now more tightly.

• Let $\Phi=\mathrm{C}_2$ and α is long. Then α can be expressed as $\alpha=\beta+\gamma$ for two *short* roots β and γ. Interchanging β and γ we can assume that $N_{\beta\gamma11}=2$. Then one has

$$x_\alpha(2\xi\zeta)=[x_\beta(\xi),x_\gamma(\zeta)]\in H.$$

Thus, we can conclude that

$$z_\alpha(2\xi\zeta,\eta) = [{}^{x_{-\alpha}(\eta)}x_\beta(\eta), {}^{x_{-\alpha}(\eta)}x_\gamma(\zeta)] =$$
$$[x_\beta(\xi)x_{-\gamma}(\pm\xi\eta)x_{\beta-\gamma}(\pm\xi^2\eta), x_\gamma(\zeta)x_{-\beta}(\pm\zeta\eta)x_{\gamma-\beta}(\pm\zeta^2\eta)] \in H.$$

One the other hand, from the second item we already know that $z_\alpha(\xi^2\zeta) \in H$. Since by assumption the ideal generated by 2ξ and ξ^2 contains ξ, no exactly the same arhument as in the second item shows that $z_\alpha(\xi,\zeta,\eta) \in H$.

• Finally, let $\Phi = \mathrm{G}_2$ and α is short. We can argue essentially in the same way as for the case of $\Phi = \mathrm{C}_2$. In a sense, it is slightly more complicated due to the fancier form of the Chevalley commutator formula. On the other hand, overall it is even easier, since we already have the necessary inclusions for *long* roots.

Again, as the first approximation, we prove that $z_\alpha(\xi^2\zeta,\eta), z_\alpha(\xi\zeta^2,\eta) \in H$. With this end, express α as $\alpha = \beta + 2\gamma$, where β is short, γ is long, and $N_{\beta\gamma 11} = 1$. Take an arbitrary $\theta \in R$. Then

$$[x_\beta(\theta\xi), x_\gamma(\zeta)] = x_{\beta+\gamma}(\theta\xi\zeta)x_\alpha(\pm\theta^2\xi^2\zeta)x_{3\beta+\gamma}(\pm\theta^3\xi^3\zeta)x_{3\beta+2\gamma}(\pm\theta^3\xi^3\zeta^2) \in H,$$

whereas

$$[x_\beta(\xi), x_\gamma(\theta\zeta)] = x_{\beta+\gamma}(\theta\xi\zeta)x_\alpha(\pm\theta\xi^2\zeta)x_{3\beta+\gamma}(\pm\theta\xi^3\zeta)x_{3\beta+2\gamma}(\pm\theta^2\xi^3\zeta^2) \in H.$$

Since the roots $3\beta+\gamma$ and $3\beta+2\gamma$ are long, from the first item we already know that the corresponding root elements belong to H, and, in fact, are already expressed as commutators $[x_\phi(\rho), x_\psi(\sigma)]$, where $\rho \in I$ and $\sigma \in J$, of two *long* root elements in $E(\mathrm{A}_2, I)$ and $E(\mathrm{A}_2, J)$. Since α is *short*, by the usual argument, conjugation by $x_{-\alpha}(\eta)$ does not move such a commutator out of H.

This means, that when comparing the above inclusions, as we did in the second item above, we can ignore all overhanging long root factors, and argue modulo the subgroup $[E(\mathrm{A}_2, I), E(\mathrm{A}_2, J)]$. This leaves us with the inclusion

$$x_\alpha(\pm(\theta^2-\theta)\xi\zeta) \in [x_\beta(\xi), x_\gamma(\theta\zeta)][x_\gamma(\zeta), x_\beta(\theta\xi)]\big[E(\mathrm{A}_2, I), E(\mathrm{A}_2, J)\big].$$

Conjugating this inclusion by $x_{-\alpha}(\eta)$, carrying the congujation inside the commutator, and observing that α is distinct from β and γ, we can in the usual way conclude that $z_\alpha(\pm(\theta^2-\theta)\xi^2\zeta,\eta) \in H$, for all $\theta \in R$. Again, since R does not have residue field of two elements, we can repeat the same argument as in the second item to derive that $z_\alpha(\xi^2\zeta,\eta) \in H$. Interchanging ξ and ζ, we see that the same holds also for $z_\alpha(\xi\zeta^2,\eta)$.

To conclude the proof it only remains to look at another short root. Namely, set $\alpha = \beta+\gamma$, for the same roots β and γ, as above. Looking at the commutator

$$[x_\beta(\xi), x_\gamma(\zeta)] = x_\alpha(\xi\zeta)x_{2\beta+\gamma}(\pm\xi^2\zeta)x_{3\beta+\gamma}(\pm\xi^3\zeta)x_{3\beta+2\gamma}(\pm\xi^3\zeta^2) \in H,$$

and carrying all factors, apart from the first one, from the right hand side to the left hand side, we get an expression of $x_\alpha(\xi\zeta)$ as a product of commutators of the form $[x_\phi(\rho), x_\psi(\sigma)]$, where either $\{\phi,\psi\} = \{\beta,\gamma\}$, or else both ϕ and ψ are long. As we

have already observed, conjugation by $x_{-\alpha}(\eta)$ does not move such a commutator out of H. This means that $z_\alpha(\xi\zeta,\eta)\in H$, as claimed. $\square$

3. Proof of Theorem 2

As a group, the relative commutator subgroup $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ is generated by the commutators $[x,y]$, where $x\in E(\Phi,R,I)$ and $y\in E(\Phi,R,J)$. By Theorem 1 we can express x as a product of generators $z_\alpha(\xi,\eta)$, where $\alpha\in\Phi$, $\xi\in I$, $\eta\in R$. Similarly, we can express y as a product of generators $z_\beta(\zeta,\theta)$, where $\beta\in\Phi$, $\zeta\in I$, $\theta\in R$.

Now, iterated application of commutator identities C1 and C2 implies that as a group $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ is generated by the conjugates ${}^g[z_\alpha(\xi,\eta),z_\beta(\zeta,\theta)]$, where $g\in E(\Phi,R)$, whereas $\alpha,\beta,\xi,\zeta,\eta,\theta$ have the same sense, as above.

Next, observe that

$${}^g\big[z_\alpha(\xi,\eta),z_\beta(\zeta,\theta)\big]={}^g\big[{}^{x_{-\alpha}(\eta)}x_\alpha(\xi),z_\beta(\zeta,\theta)\big]={}^{gx_{-\alpha}(\eta)}\big[x_\alpha(\xi),{}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)\big].$$

Since $E(\Phi,R,J)$ is normal in $E(\Phi,R)$, it follows that ${}^{x_{-\alpha}(-\eta)}z_\beta(\zeta,\theta)$ can be expressed as a product of generators $z_\gamma(\lambda,\rho)$, where $\gamma\in\Phi$, $\lambda\in J$, $\rho\in R$.

Again, iterated application of commutator identities C1 and C2 implies that $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ is generated by a smaller set of conjugates ${}^g[x_\alpha(\xi),z_\beta(\zeta,\theta)]$, where $g\in E(\Phi,R)$, whereas $\alpha,\beta,\xi,\zeta,\theta$ have the same sense, as above.

Thus, it suffices to prove that the commutators $[x_\alpha(\xi),z_\beta(\zeta,\theta)]$ are expressed as products of *conjugates* of elements from X. Now, the proof depends on the mutual position of the roots α and β.

• When $\alpha=\beta$, there is nothing to prove, since these commutators themselves belong to X. These are generators of the first type.

• When $\alpha=-\beta$, one has

$$\big[x_\alpha(\xi),z_{-\alpha}(\zeta,\theta)\big]=\big[x_\alpha(\xi),{}^{x_\alpha(\theta)}x_{-\alpha}(\zeta)\big]={}^{x_\alpha(\theta)}\big[x_\alpha(\xi),x_{-\alpha}(\zeta)\big].$$

In other words, these commutators are themselves conjugates of individual elements of X, namely, of the generators of the second type.

Thus, in the sequel we may assume that $\alpha\neq\pm\beta$.

• If α and β are strictly orthogonal, in other words, $\alpha\pm\beta\notin\Phi$, then

$$\big[x_\alpha(\xi),x_\beta(\zeta)\big]=\big[x_\alpha(\xi),x_{-\beta}(\theta)\big]=e,$$

and thus also $[x_\alpha(\xi),z_\beta(\zeta,\theta)]=e$.

• Thus, we may assume that α and β generate an irreducible root system Δ of rank 2, viz. A_2, B_2 or G_2. Thus, it suffices to prove the desired fact for Chevalley groups of rank 2.

Clearly, we may choose an order on Δ in such a way that either β or $-\beta$ is a fundamental root, whereas $\alpha\in\Delta^+$. Consider the [maximal] parabolic subgroup

$P_\beta \leq G(\Delta, R)$, corresponding to the parabolic set $\Delta^+ \cup \{\pm\beta\}$. Then $z_\beta(\zeta, \theta)$ belongs to the elementary subgroup $\langle X_\beta, X_{-\beta}\rangle$ of its Levi subgroup $L_\beta = G_\beta$. On the other hand, $x_\alpha(\xi)$ belongs to its unipotent radical

$$U_\beta = \langle U_\gamma,\ \gamma \in \Delta^+, \gamma \neq \pm\beta\rangle.$$

Recalling that by Lemma 6 the unipotent radical U_β is a normal subgroup of P_β, we see that

$$[x_\alpha(\xi), z_\beta(\zeta, \theta)] \in [U_\beta, L_\beta] \leq U_\beta \leq U(\Delta, R).$$

On the other hand, Lemma 2 implies that

$$[x_\alpha(\xi), z_\beta(\zeta, \theta)] \in G(\Delta, R, IJ).$$

Combining these inclusions, we see that

$$[x_\alpha(\xi), z_\beta(\zeta, \theta)] \in G(\Delta, R, IJ) \cap U(\Delta, R).$$

By Lemma 5 this intersection is contained in $E(\Delta, IJ)$. In particular, commutators of this type are expressed as products of elements $x_\gamma(\lambda\mu)$, where $\gamma \in \Delta^+$, $\lambda \in I$, $\mu \in J$. But this is precisely the third type of generators.

Conversely, Lemma 4 implies that these generators belong to the mixed commutator subgroup $\big[E(\Phi, R, I), E(\Phi, R, J)\big]$. This finishes the proof of Theorem 2.

4. Proof of Theorem 3 and some corollaries

It turns out, that Theorem 3 easily follows from Theorem 2, modulo other known results on relative commutator subgroups.

Proof of Theorem 3. Clearly, $X \subseteq Y$, so that by Theorem 2 the set Y generates $\big[E(\Phi, R, I), E(\Phi, R, J)\big]$ as a *normal* subgroup of $E(\Phi, R)$. Therefore, it suffices to show that conjugates of the elements of Y above generators are themselves products of elements of Y. Let $g \in Y$ and let $h \in E(\Phi, R)$. By Theorem 4 one has $g \in G(\Phi, R, IJ)$. Now applying Lemma 2 we see that

$$[h, g] \in [G(\Phi, R, IJ), E(\Phi, R)] = E(\Phi, R, IJ).$$

Thus, $hgh^{-1} = gz$, for some $z \in E(\Phi, R, IJ)$. Invoking Theorem 1 we see that hgh^{-1} is the product of g and some $z_\alpha(\xi\zeta, \eta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$. All of these factors belong to Y. This completes the proof. $\square$

A closer look at the generators in Theorem 3 shows that all of them in fact belong already to $\big[E(\Phi, I), E(\Phi, R, J)\big]$. By symmetry, we may switch the role of factors. In particular, this means that Theorem 3 implies the following curious fact.

Corollary 1. *Under assumptions of Theorem 3 one has*

$$\big[E(\Phi, I), E(\Phi, R, J)\big] = \big[E(\Phi, R, I), E(\Phi, J)\big] = \big[E(\Phi, R, I), E(\Phi, R, J)\big].$$

Let us mention another amusing corollary of our results, which was not noted before. Unlike the slightly larger commutators in the previous corollary, it is not clear, why H should be equal to $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$, but it is at least normal in the absolute elementary group.

Corollary 2. *Under assumptions of Theorem 3 the mixed commutator subgroup $\big[E(\Phi,I),E(\Phi,J)\big]$ is normal in $E(\Phi,R)$.*

Proof. By Theorem 4, we have

$$E(\Phi,R,IJ)\le\big[E(\Phi,I),E(\Phi,J)\big]\le G(\Phi,R,IJ).$$

By taking the commutator of each component of the above inclusion with $E(\Phi,R)$, the standard commutator formula implies that

$$\big[E(\Phi,R,IJ),E(\Phi,R)\big]=\big[E(\Phi,R,IJ),E(\Phi,R)\big]=E(\Phi,R,IJ).$$

It follows immediately that

$$\Big[\big[E(\Phi,I),E(\Phi,J)\big],E(\Phi,R)\Big]=E(\Phi,R,IJ)\le\big[E(\Phi,I),E(\Phi,J)\big].$$

Thus, $\big[E(\Phi,I),E(\Phi,J)\big]$ is normal in $E(\Phi,R)$. □

5. Final remarks

The relative commutator subgroups $\big[E(\Phi,R,I),E(\Phi,R,J)\big]$ considered in the present paper seem to be a very interesting class of subgroups. They occur unreasonably often in many seemingly unrelated situations.

• For the general linear group these commutator subgroups, as also the commutator subgroups of the principal congruence subgroups and full congruence subgroups

$$\big[\operatorname{GL}(n,R,I),\operatorname{GL}(n,R,J)\big],\quad \big[\operatorname{GL}(n,R,I),C(n,R,J)\big],\quad \big[C(n,R,I),C(n,R,J)\big],$$

were first considered by Hyman Bass [4], and then systematically studied by Alec Mason and Wilson Stothers [24, 21, 22, 23]. In particular, they established the formula

$$\big[\operatorname{GL}(n,R,I),\operatorname{GL}(n,R,J)\big]=[E(n,R,I),E(n,R,J)],$$

provided that $n\ge\operatorname{sr}(R)+1,3$.

• The following result, the relative standard commutator formula is the main result of You Hong [36] and the main application of the relative commutator calculus developed by us in [16].

Theorem 5 (You Hong, Hazrat—Vavilov—Zhang)**.** *Let Φ be a reduced irreducible root system, $rk(\Phi)\ge2$. Further, let R be a commutative ring, and $I,J\trianglelefteq R$ be two ideals of R. Then*

$$[E(\Phi,R,I),C(\Phi,R,J)]=[E(\Phi,R,I),E(\Phi,R,J)].$$

For the general linear group, we have three different proofs of this formula, based on decomposition of unipotents [34], localisation [19], and on level calculations [35]. Later, we generalised the last two of these proofs to Bak's unitary groups [15] and to Chevalley groups [16].

• Let us state an amazing application of Theorem 5 and our main Theorem 3, which is due to Alexei Stepanov [27], see also [10, 11].

Theorem 6 (Stepanov)**.** *Let R be a commutative ring with 1 and let $I, J \trianglelefteq R$, be ideals of R. There exists a natural number $N = N(\Phi)$ depending on Φ alone, such that any commutator*

$$[x, y], \qquad x \in G(\Phi, R, I), \quad y \in E(\Phi, R, J)$$

is a product of not more that N elementary generators listed in Theorem 3.

Quite remarkably, the bound N in this theorem does not depend either on the ring R, or on the choice of the ideals I, J. The proof is based on the method of *universal localisation* expressly developed by Stepanov [27] to eliminate any dependence on the dimension of R. Before that, even in the absolute case all known bounds depended on dimension of the ground ring, see [28, 11] and references there.

• Initially, one of the main motivations to consider relative commutator subgroups occurred in the study of subgroups of classical groups, normalised by a relative elementary subgroup. For the general linear group, this problem was studied by John Wilson, Anthony Bak, Leonid Vaserstein, Li Fuan and Liu Mulan, and the second author, and in the context of unitary groups, by Günter Habdank, the third author, and You Hong, one may find the bibliography in our survey [14] and in the conference papers [9, 10]. For Chevalley groups this problem is still not solved, and it would be a very interesting application.

• Another amazing fact, which follows from the *multiple* relative commutator formula established for classical groups in [20, 17], is that multiple commutators of relative subgroups are reduced to double such commutators. More precisely, let R be a commutative ring with 1 and let $I_i \trianglelefteq R$, $i = 1, \ldots, m$, be ideals of R. Consider an arbitrary configuration of brackets $[\![\ldots]\!]$ and assume that the outermost pairs of brackets meet between positions h and $h+1$. Then one has

$$[\![E(\Phi, R, I_1), E(\Phi, R, I_2), \ldots, E(\Phi, R, I_m)]\!] = \\ [E(\Phi, R, I_1 \ldots I_h), E(\Phi, R, I_{h+1} \ldots I_m)].$$

For Chevalley groups this result is not yet published, but it holds as stated, and can be obtained either by imitating the methods of [20, 17], or by Stepanov's universal localisation [27].

• For *finite dimensional* rings, one has even stronger *general* multiple commutator formulas, which are simultaneous generalisations of the above Mason—Stothers formula, and nilpotency of relative K_1. In these formulas, elementary subgroups on the left hand side replaced by the Chevalley groups themselves, provided that

$m > \dim(\mathrm{Max}(R))$. The definitive proofs are only published for the general linear group, see [12].

• Another very interesting occurence of the relative commutator groups is relative splitting. In general, the intersection $G(\Phi, R, I) \cap E(\Phi, R, J)$ of the principal congruence subgroup modulo one ideal with the relative elementary subgroup modulo another ideal, is very hard to describe. However, there are some instances, when one could say more. Let us state the relative splitting principle discovered by Himanee Apte and Alexei Stepanov, [3], Lemma 4. Let I be a splitting ideal of an associative ring R, $R = I \oplus A$, and let $J \trianglelefteq R$ be an ideal of R generated by an ideal $K \trianglelefteq A$. Then

$$G(\Phi, R, I) \cap E(\Phi, R, J) = [E(\Phi, R, I), E(\Phi, R, J)].$$

• Finally, let us mention that it seems that there is a close connection between the relative commutator subgroups and excision kernels. We refer to [10] for the references on multiple relativisation and excision kernels, and some further discussion.

In the present paper we considered only the usual relative groups, defined in terms of a *single* ideal. Actually, for multiply laced systems relative subgroups should be parametrised not by the usual ideals, but by *admissible pairs* (A, B), as defined by Eiichi Abe [1]. In such a pair, the ideal A parametrises elementary generators for short roots Φ_s, whereas the *additive subgroup* B parametrises elementary generators for long roots Φ_l. Moreover, $A_p \leq B \leq A$, where $p = 2$ for $\Phi = \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4$ and $p = 3$ for $\Phi = \mathrm{G}_2$. Here A_p is the ideal generated by $p\xi$ and ξ^p, for all $\xi \in A$. In all non-symplectic cases B is also an ideal of R, but in the symplectic case it is a *Jordan ideal*. For classical groups, admissible pairs are precisely the special case of form ideals as defined the same year by Anthony Bak.

In the non simply laced case, the genuine relative elementary subgroups, which occur in the classification of normal subgroups of $G(\Phi, R)$, are parametrised by admissible pairs, rather than individual ideals, and are defined as follows:

$$E(\Phi, R, A, B) = \langle x_\alpha(\xi), \alpha \in \Phi_s, \xi \in A;\ x_\beta(\zeta), \beta \in \Phi_l, \zeta \in B\rangle^{E(\Phi,R)}.$$

There is an analogue of Theorem 1 in this case, obtained by Michael Stein [25], and explicitly stated by Eiichi Abe [2].

Theorem 7 (Stein—Abe). *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I be an ideal of a commutative ring R. Then as a group $E(\Phi, R, A, B)$ is generated by the elements of the form $z_\alpha(\xi, \eta)$, where $\xi \in A$ for $\alpha \in \Phi_s$ and $\xi \in B$ for $\alpha \in \Phi_l$, while $\eta \in R$.*

Thus, the following problem naturally suggests itself.

Problem 1. *Generalise Theorems 2 and 3 to the case of the relative commutator subgroups $\big[E(\Phi, R, A, B), E(\Phi, R, C, D)\big]$, for two admissible pairs (A, B) and (C, D).*

Actually, in the case, where $\Phi = \mathrm{B}_2, \mathrm{G}_2$ and the ring has residue fields $\mathbb{F}_2$ of two elements, the situation is even more complicated. Douglas Costa and Gordon

Keller studied the normal structure of $\mathrm{Sp}(4,R)$ and $G_2(R)$ in their remarkable papers [6, 7] and discovered that in these cases relative subgroups should be parametrised by *radices*, rather than by individual ideals, or admissible pairs.

Problem 2. *For the cases $\Phi = \mathrm{B}_2$ and $\Phi = \mathrm{G}_2$ generalise Theorems 2 and 3 to the relative commutators of elementary subgroups, defined in terms of radices.*

We refer the interested reader to our papers cited above, and to our conference papers [9, 10, 11, 18], where one can find many further details and unsolved problems concerning relative commutator subgroups.

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